MAXIMAL INEQUALITIES IN BILATERAL GRAND LEBESQUE SPACES OVER UNBOUNDED MEASURE

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Abstract

In this paper non-asymptotic exact rearrangement invariant norm estimates are derived for the maximum distribution of the family elements of some rearrangement invariant (r.i.) space over unbounded measure in the entropy terms and in the terms of generic chaining.

We consider some applications in the martingale theory and in the theory of Fourier series.

Key words: Generic chaining, rearrangement invariant spaces, metric entropy, natural distance, natural space, moment, Grand Lebesgue Spaces, fundamental function, moment, martingales.

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1. Introduction. Notations. Statement of problem.

Let (X, Σ, μ) be a measurable space with non-trivial measure $\mu: \ \exists A \in \Sigma, \mu(A) \in (0, \mu(X))$.

We will assume that $\mu(X) = \infty$ and that the measure μ is $\sigma-$ finite and diffuse: $\forall A \in \Sigma, 0 < \mu(A) < \infty \ \exists B \subset A, \mu(B) = \mu(A)/2$.

Let also $T = \{t\}$ be *arbitrary* set and $Y = Y(t, x) = Y(t)$ be some function of a variables t and x such that for all the values $t \in T$ the function $Y = Y(t, x)$ is measurable as a function on x and is separable.

Definition 1. The function $Y = Y(t, x)$ is called separable relatively the variable t; $t \in T$, if there exists a countable subset $\tilde{T}$ of a set $T: \ \tilde{T} = \{t_1, t_2, \ldots\} \subset T$ such that for arbitrary closed set Q on the space $R = R^1$

$$\cap_{t \in \tilde{T}} \{x : Y(t) \in Q\} \sim \cap_{t \in T} \{x : Y(t) \in Q\}. \tag{1.1}$$

Here and further the set equivalence $A \sim B, \ A, B \subset X$ means that both the sets A and B are measurable: $A \in \Sigma, \ B \in \Sigma$ and

$$\mu\{(A \setminus B) \cup (B \setminus A)\} = 0.$$

As a rule, the set $\tilde{T}$ is enumerable dense subset of T relatively some distance (or semi-distance) $r = r(t,s), \ t,s \in T$ on the set T. We will call in this case the subset $\tilde{T}$ the *separante* of the set T and will write

$$\tilde{T} = sep(T,r). \tag{1.2}$$

For example, if the function $Y = Y(t,\cdot)$ is continuous a.e. on the variable t in the distance r, and the metric space (T,r) is separable, then $Y(t,x)$ is separable. Further, if the set T is the union of some sequence subsets $S_m, m = 1,2,\ldots,M, \ M \le \infty$ of the set T

$$T = \cup_{m=1}^{M} S_m$$

and the function Y is separable on the sets S_m, then $Y = Y(t,x)$ is separable on the set T.

Let us define

$$\overline{Y} = \overline{Y}(x) = \sup_{t \in T} Y(t,x). \tag{1.3}$$

It is easy to verify, as in the theory of random processes, that if the function $Y = Y(t,x)$ is separable, then $\overline{Y}(x)$ is measurable function on the variable x.

Further we will assume that our function $Y = Y(t,x)$ is separable over some suitable dense set $\tilde{T}$.

Let also G be some rearrangement invariant (r.i.) space with a norm $||\cdot||G$ over our triplet (X,Σ,μ), for instance, $L_p = L_p(X,\Sigma,\mu)$, Orlicz, Marcinkiewicz, Lorentz or Grang Lebesque spaces etc.

Our aim is obtaining the $G -$ norm estimation for $\overline{Y}$: $||\ \overline{Y}\ ||G$ through some simple rearrangement invariant parameters of source function $Y(t,x)$.

The important examples of these parameters are: the maximal value

$$\sigma = \sigma(G) \stackrel{def}{=} \sup_{t \in T} ||Y(t,\cdot)||G \tag{1.4}$$

and the so-called $G -$ distance (more exactly, semi-distance) $d_G = d_G(t,s)$ on the set T :

$$d_G = d_G(t,s) \stackrel{def}{=} ||\ Y(t,\cdot)\ -\ Y(s,\cdot)\ ||G. \tag{1.5}$$

Recall that the semi-distance $d = d(t,s), \ s,t \in T$ is, by definition, non-negative symmetrical numerical function, $d(t,t) = 0, \ t \in T$, satisfying the triangle inequality, but the equality $d(t,s) = 0$ does not means (in general case) that $s = t$.

It is evident that if $\sigma(G) < \infty$, then $d_G(t,s) \le 2\sigma(G)$.

Notice that the case $\mu(X) = 1$ (the probabilistic case) is well investigated in the theory of random fields, see, for example,[1], [2], [3], [4], [5], [6], [41] etc. The

obtained there results may be used here as illustration of precision of estimations of this article.

We will use widely further the notion of *fundamental function* $\phi(G, \delta), \ \delta \in (0, \infty)$ of the r.i. space G. Recall that by definition

$$\phi(G, \delta) = || \ I(A) \ || G, \ \mu(A) = \delta$$

and $I(A) = I(A, x) = 1, x \in A, \ I(A) = I(A, x) = 0, x \notin A.$

This notion play a very important role in the theory of interpolation of operators, theory of Fourier series, theory of approximation etc. See, for example, [30], [23], [42] etc.

Let the set T relatively some semi-distance $r = r(t, s)$ be precompact set. We denote by $N = N(T, r, \epsilon)$ the minimal number of closed $r -$ balls $B(r; t_j, \epsilon), \ t_j \in T$ with the center t_j and the radius $\epsilon, \ \epsilon > 0$:

$$B(r; t_j, \epsilon) = \{t, \ t \in T, \ r(t, t_j) \le \epsilon\}$$

covering the set T :

$$N(T, r, \epsilon) \stackrel{def}{=} \min\{K, \ \exists \{t_j\}, j = 1, 2, \ldots, K; \ t_j \in T, \ T \subset \cup_{j=1}^{K} B(r; t_j, \epsilon)\}.$$

The (natural) logarithm of $N(T, r, \epsilon)$: $\ H(T, r, \epsilon) = \log N(T, r, \epsilon)$ is called entropy of T in the distance r, and the value (finite or infinite)

$$\kappa = \kappa_r(T) \stackrel{def}{=} \overline{\lim}_{\epsilon \to 0+} \frac{H(T, r, \epsilon)}{|\log \epsilon|}$$

is called the *dimension* T in the distance r :

$$\kappa_r(T) = dim_r(T).$$

2. Grand Lebesque spaces.

We define as usually for arbitrary measurable function $f : X \to R^1$

$$\mathbf{E} f = \int_X f(x) \mu(dx); \ p \ge 1 \ \Rightarrow$$

$$|f|_p = \mathbf{E}^{1/p} \left(|f|^p\right) = \left(\int_X |f(x)|^p \ \mu(dx) \right)^{1/p};$$

$L_p = L(p) = L(p; X, \mu) = \{f, |f|_p < \infty\}.$

Let $a = const \ge 1, b = const \in (a, \infty]$, and let $\psi = \psi(p) = \psi(p; a, b)$ be some strong positive: $\psi(p) \ge 1$ bounded in each open subinterval $(c, d), \ a < c < d < b$ logarithmical convex on the *open* interval (a, b) function.

We will denote the set of all such a function by Ψ : $\ \Psi = \Psi(a, b) = \{\psi\} = \{\psi(\cdot; a, b)\}.$

Definition 2. The space $BGL(\psi) = G(\psi) = G(X,\psi) = G(X,\psi,\mu) = G(X,\psi,\mu,a,b)$ (Bilateral Grand Lebesque space) consist on all the measurable functions $f: X \to R$ with finite norm

$$||f||G(\psi) \stackrel{def}{=} \sup_{p \in (a,b)} \left[|f|_p / \psi(p) \right]. \tag{2.1}$$

We can define formally in the case $a = b \in [1,\infty)$ $G(\psi) = L_a$.

Suppose that there exist a *pair* of numbers $(a,b), \ 1 \le a < b \le \infty$ such that

$$\forall p \in (a,b) \ \Rightarrow \ |Y(t,\cdot)|_p < \infty$$

and such that

$$\forall \Delta > 0 \ \Rightarrow \ \sup_{t \in T} |Y(t,\cdot)|_{a \ - \ \Delta} = \infty \tag{2.2}$$

and

$$\forall \Delta > 0 = \sup_{t \in T} |Y(t,\cdot)|_{b+\Delta} = \infty \tag{2.3}$$

where in the case $a = 1$ the condition (2.2) is absent and in the case $b = \infty$ the condition (2.3) is absent.

Then we can define the following *natural choice* of a function $\psi_0(p)$ as follows:

$$\psi_0(p) \stackrel{def}{=} \sup_{t \in T} |Y(t,\cdot)|_p. \tag{2.4}$$

The spaces $G(\psi), \ \psi \in U\Psi$ are non-trivial: arbitrary bounded $\sup_x |f(x)| < \infty$ measurable function $f: X \to R$ with finite support: $\mu(supp \ |f|) < \infty$ belongs to arbitrary space $G(\psi)$.

We denote as usually $supp \ \psi = \{p: \ |\psi(p)| < \infty\}$.

The detail investigation of these spaces see, for example, in [14], [15], [17], [18], [34], [42] etc.

It is knows (see [42]) that the BGL spaces in general case does not coincide with classical r.i. spaces: Lorentz, Marcinkiewicz, Orlicz spaces. It is obvious that BGL spaces does not coincide with recently appeared Grand Orlicz, modular and variable Lebesgue spaces, as long as both the last spaces are not, in general case, rearrangement invariant (see [37], [38], [39]).

The BGL spaces are used, for example, in the theory of probability [2]- [7], [8] - [10], [42]; theory of PDE [14], [15], functional analysis [11], [12], [42], theory of Fourier series [23], [30], theory of martingales [14], [15], [16] etc.

If we choose as the r.i. space G the space $G(\psi_0)$, then $\sigma(G(\psi_0)) = 1$; and we can introduce the so-called *natural* distance

$$d_0(t,s) \stackrel{def}{=} ||Y(t,\cdot) \ - \ Y(s,\cdot)||G(\psi_0), \ t,s \in T.$$

This approach in the probabilistic case was introduced by [36] for Gaussian random fields; more general case was considered in [4].

The fundamental function of BGL spaces may be calculated by the formula:

$$\phi(G(\psi),\delta) = \sup_{p \in (a,b)} [\delta^{1/p}/\psi(p)].$$

Many examples of $G(\psi)$ spaces and its fundamental functions see in [42]. As a particular case $G(\psi)$ space may coincide with arbitrary exponential Orlitzs space.

The spaces $G(\psi, a, b)$ are non-separable and non-reflexive ([42]), but they satisfy the Fatou property. Namely, the following property about these spaces is true.

Proposition 1. The $G(\psi)$ space satisfies the Fatou property.

Proof. Recall at first that the Fatou property of some r.i. space G over source triplet (X, Σ, μ) denotes that for arbitrary non-increasing sequence of non-negative functions $\{f_n\} = \{f_n(x),\ x \in X\}$ belonging to the space G and such that as $n \uparrow \infty$

$$f_n(x) \uparrow f(x),\ \sup_n ||f_n||G < \infty \tag{2.5}$$

it follows

$$||f_n||G \uparrow ||f||G. \tag{2.6}$$

Let $G = G(\psi)$ and let the sequence of measurable functions $\{f_n\} = \{f_n : X \to R\}$ satisfies the condition (2.5). As long as the space $L_p(X, \mu)$ satisfies the Fatou property, we have:

$$\sup_n ||f_n||G(\psi) = \sup_n \sup_{p \in (a,b)} [|f_n|_p/\psi(p)] =$$

$$\sup_{p \in (a,b)} \sup_n [|f_n|_p/\psi(p)] = \sup_{p \in (a,b)} [|f|_p/\psi(p)] = ||f||G(\psi),$$

Q.E.D.

As a simple consequence: it follows from theorem of Mityagin - Kalderon that the space $G(\psi)$ is interpolation space between spaces $L_1(X, \mu)$ and $L_\infty(X, \mu)$. See in detail [11], [12].

3. Main results.

A. Generic chaining theory in our case.

Now we recall, modify and rewrite some definition from the generic chaining theory, belonging to X.Fernique [1] and M.Talagrand [6] - [10].

Let $(G, || \cdot ||G$ be some r.i.space over (X, Σ, μ) and let

$$\tilde{T} = sep(T, d_G).$$

Definition 3. The generic chaining W is, by definition, the partition of the set $\tilde{T}$ into a sequence of finite subsets $\{Q_k\}$:

$$\tilde{T} = \cup_{k=0}^{\infty} Q_k,$$

where $|Q_k| \stackrel{def}{=} card\ (Q_k) < \infty$. Notation: $W = \{Q_k\}$.

Without loss of generality we can and will assume that $Q_0 = \{t_0\}$, where

$$\sigma(G) = \sup_{t \in T} ||Y(t, \cdot)||G = ||Y(t_0, \cdot)||G.$$

For any element $t \in T$ we denote arbitrary, but fixed (non-random) element $\pi_k(t)$ of a subset Q_k such that

$$d_G(t, \pi_k(t)) = \min_{s \in Q_k} d_G(t, s). \tag{3.1}$$

Thus,

$$||Y(t, \cdot) \ - \ Y(\pi_k(t), \cdot)||G \le d_G(t, \pi_k(t)). \tag{3.2}$$

Let us denote for some partition $W = \{Q_k\} = \{Q(k)\}$

$$\Lambda(T, G, W) = \sum_{k=0}^{\infty} || \max_{t \in Q_k} (Y(\pi_k(t), \cdot) - Y(\pi_{k-1}(t), \cdot)||G.$$

Proposition 2.

$$||\overline{Y}||G \le \inf_W \Lambda(T, G, W). \tag{3.3}$$

Proof is very simple. Let R be arbitrary partition. Since the function $Y = Y(t, x)$ is presumed to be separable, we have a.e.:

$$\overline{Y} = \lim_{M \to \infty} \max_{t \in \cup_{k=1}^M Q(k)} Y(t, x) \le$$

$$\lim_{M \to \infty} \sum_{k=0}^{M} \max_{t \in Q_k} (Y(\pi_k(t), x) \ - \ Y(\pi_{k-1}(t), x)).$$

We find using the triangle inequality for the $G-$ norm

$$||\overline{Y}||G \le \Lambda(T, G, W). \tag{3.4}$$

Since the partition W is arbitrary, we get to the (3.3) after the minimization over W.

Following, we need to estimate the $G-$ norm for the maximal value of finite set of a functions. At first we use the so-called Pizier technique.

B. (Finite case). We suppose here that the set T is finite: $T = \{t_1, t_2, \ldots, t_m\}$; on the other words, $card(T) = m < \infty$, and assume that for some $p \in [1, \infty)$

$$\max_{j=1,2,\ldots,m} |Y(t_j, \cdot)|_p < \infty.$$

Proposition 3.

We provide the following generalization of famous Piziers [10] inequality:

$$|\overline{Y}|_p \le \max_{j=1,2,\ldots,m} |Y(t_j, \cdot)|_p \cdot m^{1/p}. \tag{3.5}$$

Proof. Indeed, assume for simplicity $\mid Y(t_j) \mid_p \le 1$. We get:

$$[\overline{Y}]^p = \max_{j=1,2,,m} [Y(t_j,\cdot)]^p \le \sum_{j=1}^m [Y(t_j,\cdot)]^p;$$

$$|\overline{Y}|_p^p \le \sum_{i=1}^m |Y(t_i,\cdot)|_p^p \le m.$$

C. (Generalization of finite case).
Let ψ, ζ, ν be three function from the set $\Psi(a,b)$ such that

$$\zeta(p) = \psi(p)\ \nu(p),\ p \in (a,b).$$

We suppose again here that the set T is finite: $T = \{t_1, t_2, \ldots, t_m\}$; and assume that for some $p \in [1, \infty)$

$$\max_{j=1,2,\ldots,m} |Y(t_j,\cdot)|_p < \infty.$$

Proposition 4.

$$||\overline{Y}||G(\zeta) \le \max_{i=1,2,,m} ||f_i||G(\psi) \cdot \phi(G(\nu), m). \tag{3.6}$$

Proof. We may use the inequality (3.5), estimating the values $|f_i|_p$ as

$$|f_i|_p \le ||f_i||G(\psi) \cdot \psi(p),$$

on the basis of definition the norm in the $G(\psi)$ space. We have:

$$|\overline{Y}|_p \le \max_{i=1,2,,m} ||f_i||G(\psi) \cdot \psi(p) \cdot m^{1/p}.$$

Dividing by $\zeta(p)$ and tacking supremum over $p \in (a,b)$, we receive:

$$||\overline{Y}||G(\zeta) \le \max_{i=1,2,,m} ||f_i||G(\psi) \cdot \sup_{p \in (a,b)} \frac{m^{1/p}}{\nu(p)} =$$

$$\max_{i=1,2,,m} ||f_i||G(\psi) \cdot \phi(G(\nu), m),$$

Q.E.D.

Let now and further θ be some *fixed* number inside the interval (0, 1), for example, $\theta = 1/2$ or $\theta = 1/e$. We suppose for some $p \ge 1$

$$\sup_{t \in T} |Y(t,\cdot)|_p < \infty,$$

and denote

$$d_p(t,s) \stackrel{def}{=} \mid Y(t,\cdot)\ -\ Y(s,\cdot) \mid_p.$$

We consider here as the set Q_k and consequently the partition W in (3.3) the minimal θ^k set of the space T under the distance d_p; recall that the quantity of its element is equal to $N(T, d_p, \theta^k)$.

Proposition 5.

$$| \overline{Y} |_p \le \sum_{k=1}^{\infty} \theta^{k-1} N^{1/p}(T, d_p, \theta^k). \tag{3.7}$$

Proof. This proposition follows immediately from proposition 2 and our generalization of Pizier inequality (3.5):

$$| \max_{t \in Q(k)} (Y(\pi_k(t), \cdot) \ - \ Y(\pi_{k \ - \ 1}(t), \cdot))|_p \le \theta^{k-1} N^{1/p}(T, d_p, \theta^k)$$

after summing over k.

Remark 1. We can rewrite the inequality (3.7) as follows:

$$| \overline{Y} |_p \le \inf_{\theta \in (0,1)} \sum_{k=1}^{\infty} \theta^{k-1} N^{1/p}(T, d_p, \theta^k).$$

Formulation of main result.

Let as in the proposition 4 ψ, ζ, ν be three function from the set $\Psi(a, b)$ Fixing some pair $a, b : \ 1 \le a < b \le \infty$ and a three functions $\zeta(\cdot), \psi(\cdot), \nu(\cdot)$ from the space $\Psi(a, b)$ such that

$$\zeta(p) = \psi(p) \ \nu(p), \ \ p \in (a, b),$$

we assume that

$$\sup_{t \in T} || \ Y(t, \cdot) \ || G(\psi) < \infty,$$

and denote

$$d_\psi(t, s) = || \ Y(t, \cdot) - Y(s, \cdot) \ || G(\psi).$$

For example, $\psi(p)$ may coincide with the natural function $\psi_0(p)$.

We consider in this section as the set Q_k and consequently the partition W in (3.3) the minimal θ^k – set of the space T under the distance d_ψ; recall that the quantity of its element is equal to $N(T, d_\psi, \theta^k)$.

Theorem 1.

$$|| \ \overline{Y} \ || G(\zeta) \le \inf_{\theta \in (0,1)} \sum_{k=1}^{\infty} \theta^{k-1} \phi\left(G(\nu), N(T, d_\psi, \theta^k) \right). \tag{3.8}$$

Proof is at the same as in the proposition 5; instead the Pizier inequality (3.5) we use its generalization (3.6).

Note that it follows from conclusion of Theorem 1 the *continuity* of $Y(t)$ with probability one in the semi-distance d_ψ :

$$\mu\{x : Y(\cdot, x) \notin C\,(T, d_\psi)\} = 0;$$

$C(T, d)$ denotes as usually the space of all continuous with respect to the semi-distance d functions $f : T \to R$.

The conditions of theorem 1 in the probabilistic case $\mu(X) = 1$ are equivalent to the so-called condition of the convergence of the majoring integral, see [7], [8].

Examples.

Example 1. Let under the conditions of theorem 1 for all values $\epsilon \in (0, \theta)$ and for some $\kappa = const > 0$

$$N(T, d_\psi, \epsilon) \le C\ \epsilon^{-\kappa}. \tag{3.9}$$

Denote for the values $p > \max(\kappa, 1)$

$$\psi^{(\kappa)}(p) = \psi(p) \cdot \frac{p}{p - \kappa}. \tag{3.10}$$

As long as

$$N(T, d_p, \theta^k) \le N(T, d_\psi, \theta^k / \psi(p)),$$

we obtain after some calculations using the result (3.7) of the proposition 5:

$$|\overline{Y}|_p \le \psi(p) + [\psi(p)]^{\kappa/p} \sum_{k=1}^{\infty} \theta^{k(1-\kappa/p)} \le$$

$$\psi(p) + C[\psi(p)]^{\kappa/p} \cdot \left[\theta^{\kappa/p}\ -\ \theta)^{-1}\right] \le$$

$$C\psi(p) \left[1 + (\theta^{\kappa/p}\ -\ \theta)^{-1}\right] \le C\psi^{(\kappa)}(p),\ C = const. \tag{3.11}$$

Therefore, under considered conditions

$$||\overline{Y}||G\left(\psi^{(\kappa)}\right) \le C \sup_{t \in T} ||Y(t, \cdot)||G(\psi). \tag{3.12}$$

Since

$$||\ \overline{Y}\ ||G(\psi) \ge \sup_{t \in T} ||Y(t, \cdot)||G(\psi),$$

we conclude that the estimation (3.12) is exact up to multiplicative constant in the case if $\psi(\cdot) \in \Psi(a, b),\ \zeta(p) = \psi(p)$, where $\kappa < a$; the last condition is satisfied automatically if $\kappa < 1$.

In the case if for all values $\epsilon < \theta$

$$N(T, d_\psi, \epsilon) \le C\ \epsilon^{-\kappa(1)}\ |\log \epsilon|^{-\kappa(2)}, \tag{3.13}$$

$\kappa(1) = const > 0, \kappa(2) = const < \kappa(1)$, we obtain after some calculations denoting for the values $p > \kappa(1),\ p \in (a, b)$

$$\psi_{\kappa(1),\kappa(2)}(p) = \left[\frac{p}{p\ -\ \kappa(1)}\right]^{1-\kappa(2)/\kappa(1)} \cdot \psi(p):$$

$$||\overline{Y}||G\left(\psi_{\kappa(1),\kappa(2)}\right) \le C\ \sup_{t\in T} ||Y(t,\cdot)||G(\psi). \tag{3.14}$$

In the case if the condition (3.13) is satisfied and $\kappa(1) = const > 0, \kappa(2) = \kappa(1)$, we conclude denoting

$$\psi_{l,\kappa(1),\kappa(2)}(p) = \left|\frac{\log(p\ -\ \kappa(1))}{\log(p)}\right|_{+} \cdot \psi(p),$$

$z_+ = \max(z,1):$

$$||\overline{Y}||G\left(\psi_{l,\kappa(1),\kappa(2)}\right) \le C\ \sup_{t\in T} ||Y(t,\cdot)||G(\psi). \tag{3.15}$$

Finally, in the case if the condition (3.13) is satisfied and $\kappa(1) = const > 0, \kappa(2) > \kappa(1)$, we conclude:

$$||\overline{Y}||G(\psi) \le C\ \sup_{t\in T} ||Y(t,\cdot)||G(\psi). \tag{3.16}$$

The estimations (3.14), (3.15), (3.16) it follow from Theorem 1 and the elementary inequalities (3.17.1), (3.17.2), (3.17.3), where we denote

$$S_\beta(q) = \sum_{k=1}^{\infty} q^k\ k^\beta,\ q \in [1/2,1),\ \beta = const:$$

$$\beta > -1\ \Rightarrow\ S_\beta(q) \le C(\beta)\ (1\ -\ q)^{-1-\beta}; \tag{3.17.1}$$

$$\beta = -1\ \Rightarrow\ S_\beta(q) \le C\ |\log(1\ -\ q)|; \tag{3.17.2}$$

$$\beta < -1\ \Rightarrow\ S_\beta(q) \le C(\beta). \tag{3.17.3}$$

Example 2. Exponential Orlicz spaces.
We consider here as a space G a so-called exponential Orlicz spaces.

Definition 3. We introduce the $N(a,\beta) = N(a,\beta;u), a \ge 1, \beta > 0$ as an Orliczs function such that

$$u \to 0\ \Rightarrow\ N(a,\beta;u) \sim C_1|u|^a;$$

$$|u| \to \infty\ \Rightarrow N(a,\beta;u) = \exp\left(C_2|u|^{1/\beta}\right).$$

The correspondent Orlicz space defined over source triple with $N-$ Orlicz function $\Phi(u) = \Phi(a,\beta;u)$ will denoted as $Or(a,\beta)$ and the norm of a (measurable) function $f: X \to R$ in this space will denoted as

$$||f||G(a,\beta) = ||f||Or(a,\beta) = ||f||Or(\Phi(a,\beta;\cdot). \tag{3.18}$$

Let $a = const \ge 1, \beta(1), \beta(2) = const,\ 0 < \beta(1) < \beta(2) < \infty$. Suppose that

$$\sup_{t \in T} ||Y(t,\cdot)||G(a,\beta(1)) < \infty$$

and introduce a distance $d_{a,\beta(1)}(t,s)$ by the formula

$$d_{a,\beta(1)} = d_{a,\beta(1)}(t,s) = ||Y(t,\cdot)\ -\ Y(s,\cdot)||G(a,\beta(1)).$$

We assert:

$$||\overline{Y}||G(a,\beta(2)) \le C\ \sup_{t \in T} ||Y(t,\cdot)||G(a,\beta(1)) \times$$

$$\inf_{\theta \in (0,1)} \sum_{k=1}^{\infty} \theta^{k\ -\ 1} H^{\beta(2)-\beta(1)} \left(T, d_{a,\beta(1)}, \theta^k \right). \tag{3.19}$$

Recall that $H(T,d,\epsilon) = \log N(T,d,\epsilon)$.

The proof of (3.19) it follows from theorem 1 and from the fact that the space $Qr(a,\beta)$ coincides up to the norm equivalence with some

$G(\psi) = G(\psi; a, \infty)$ space:

$$||f||G(a,\beta) = ||f||Or(a,\beta) \asymp \sup_{p \ge a} \frac{|f|_p}{p^{\beta}}.$$

See for example [23], [42] where is formulated and proved more general assertion.

Note that the inequality (3.19) is alike to the famous Dudley condition for continuity of Gaussian random field [36].

Note also that the condition

$$\inf_{\theta \in (0,1)} \sum_{k=1}^{\infty} \theta^{k\ -1}\ H^{\beta(2)-\beta(1)} \left(T, d_{a,\beta(1)}, \theta^k \right) < \infty \tag{3.20}$$

is satisfied if for example

$$dim(d_{a,\beta(1)}, T) < \infty.$$

4. Generalization on the moment rearrangement spaces.

Let $(G,\ ||\cdot||G)$ be some r.i. space defined over our triplet (X,Σ,μ). We reproduce in this section the notion of the so-called *moment rearrangement invariant (m.r.i.)* space from [29] and consider the generalization of maximal inequality on m.r.i. spaces.

Definition 4.

We will say that the r.i. space $G = G(m) = G_m$ with the norm $||\cdot||G = ||\cdot||G(m)$ is moment rearrangement invariant space, briefly: m.r.i. space, or
$G = G(m) = (G,\ ||\cdot||G) \in m.r.i.$, if there exist a real constants $a, b; 1 \le a < b \le \infty$,

and some rearrangement invariant norm $< \cdot >$ defined on the space of a real functions defined on the interval (a,b), not necessary to be finite on all the functions, such that

$$\forall f \in G \ \Rightarrow ||f||G =< \ h(\cdot) \ \ >, \ h(p) = |f|_p. \tag{4.1}$$

We will write for considered m.r.i. spaces $(G, \ ||\cdot||G)$

$$(a,b) \stackrel{def}{=} supp(G),$$

moment support; not necessary to be uniquely defined.

There are many r.i. spaces satisfied the condition (4.1) aside from $G(\psi)$ spaces: exponential Orlicz's spaces, Marcinkiewicz spaces, interpolation spaces (see [29], [33], [35]).

In the article [32] are introduced the so-called $Q(p,\alpha)$ spaces consisted on all the measurable function $f: T \to R$ with finite norm

$$||f||_{p,\alpha} = \left[\int_1^\infty \left(\frac{|f|_x}{x^\alpha} \right)^p \ \nu(dx) \right]^{1/p},$$

where ν is some Borelian measure.

Astashkin in [33] proved that the space $Q(p,\alpha)$ in the case $T = [0,1]$ and $\nu = m, \ m$ is Lebesgue measure coincides with the Lorentz $\Lambda_p(\log^{1-p\alpha}(2/s))$ space. Therefore, both this spaces are m.r.i. spaces.

Since for arbitrary real-valued continuous function f defined on the set $[0,1]$

$$||f||C[0,1] = \sup_{t\in[0,1]} |f(t)| = \lim_{p\to\infty} |f|_p = \sup_{p\in[1,\infty)} \ | \ |f|_p \ |,$$

the space $C[0,1]$ is m.r.i. space with $supp(C[0,1]) = [1,\infty)$ or equally, e.g., $supp(C[0,1]) = [3,\infty)$.

But there exist rearrangement invariant spaces without m.r.i. property [29].

Let $G = G_m$ be some m.r.i. space and suppose for all values $p \in (a,b)$

$$\sup_{t\in T} |Y(t,\cdot)|_p < \infty.$$

Denote as in the section 3

$$d_p(t,s) = | \ Y(t,\cdot) \ - \ Y(s,\cdot)|_p.$$

Proposition 6.

We denote also

$$g(p) = \inf_{\theta\in(0,1)} \sum_{k=1}^\infty \theta^{k-1} N^{1/p}(T, d_p, \theta^k).$$

It follows from the definition of m.r.i. spaces (4.1) and from the proposition 5 that

$$|| \ \overline{Y} \ ||G(m) \ \le < g > . \tag{4.2}$$

5. Application to the martingale theory over the spaces with infinite measure.

Let $(S_n, F_n) = (S(n), F(n))$ be a martingale, i.e. a monotonically non decreasing sequence of $F_n -$ sigma - subalgebras of Σ and $F_n = F(n)$ measurable functions S_n such that $\mathbf{E} S_{n+1}/F_n = S_n$ **a.e.**.

We define formally $S(0) = S_0 = 0; \ F(0) = F_0 = \{\emptyset, X\}$.

In this section we will use also the probabilistic notations

$$\mathbf{Var}\ f = \mathbf{Var}(f) = \mathbf{E}(f - \mathbf{E}f)^2 = |f - \mathbf{E}f|_2^2$$

and notation $\mathbf{E}f/F$ for the conditional expectation.

Denote

$$\sigma(n) = [\mathbf{Var}(S_n)]^{1/2}$$

and suppose the function $n \to \sigma(n)$ be *regular* varying:

$$\sigma(n) = n^{\gamma} L(n), \gamma = const > 0,$$

where $L = L(n)$ is *slowly* varying as $n \to \infty$:

$$\forall C > 0 \ \Rightarrow \ \lim_{n \to \infty} L(Cn)/L(n) = 1.$$

It is obvious that

$$\sigma^2(n) = \sum_{k=1}^{n} ||S(k) - S(k - 1)||_2^2.$$

The $L_p -$ theory of conditional expectations and theory of martingales in the case $\mu(X) = \infty$ and some its applications see, for example, in the book [24], pp. 330 - 347; see also [25], [26].

The Orlicz's norm estimates for martingales are used in the modern non - parametrical statistics, for example, in the so - called regression problem ([4], [42] etc).

We recall here the famous inequality of Doob:

$$p > 1 \ \Rightarrow \left| \sup_{n \in [1,N]} |S_n| \right|_p \le \sup_{n \in [1,N]} \left[|S_n|_p \ p/(p-1) \right], \tag{6.1}$$

where $N = 1, 2, \ldots, \infty.$

Let $v = v(n)$ be some non-decreasing positive deterministic function, $v(n) \to \infty$ as $n \to \infty$. We purpose that for some $\psi \in \Psi(a, b)$

$$\sup_n ||S(n)/\sigma(n)||G(\psi) < \infty. \tag{6.2}$$

We will obtain in this section using (6.1) the rearrangement norm estimations for the value

$$\tau = || \sup_n [S(n)/(v(n) \ \sigma(n))] \, ||G(\psi_1), \tag{6.3}$$

where at $p > 1$

$$\psi_1(p) = p\ \psi(p)/(p\ -\ 1).$$

In the entropy and generic chaining terms in the probabilistic case $\mu(X) = 1$ this estimations are obtained in [13], [16], [40].

Theorem 2. Let $v = v(n)$ be such that

$$\sum_{n=1}^{\infty} 1/v\,(2^n) < \infty. \tag{6.4}$$

Then

$$||\tau||G(\psi_1) \le C\ \sup_n ||S(n)/\sigma(n)||G(\psi). \qquad (6.5).$$

Proof. We intend to use the inequality (3.3), where instead Pizier assertion we will use the Doob's inequality.

Choosing the *partition* over the closed intervals $W = \{[A(k), A(k+1) - 1]\} = \{[A(k), B(k)] = \{Q(k)\}\}$ of a view:

$$Q(k) = [A(k), B(k)] = [2^{k-1}, 2^k\ -\ 1],\ k = 1, 2, \ldots.$$

Suppose for simplicity

$$\sup_n ||S(n)/\sigma(n)||G(\psi) = 1.$$

Let us denote

$$\tau(k) = \max_{m \in Q(k)} |S(m)/(\sigma(m)\ v(m)|;$$

then

$$|\tau|_p \le \sum_k |\tau(k)|_p.$$

Further,

$$|\tau(k)|_p = \left| \max_{m \in Q(k)} \frac{|S(m)|}{\sigma(m)\ v(m)} \right|_p \le |\max_{m \in Q(k)} |S(m)|/(v(A(k))\ \sigma(A(k))))|_p \le$$

$$\frac{p}{p\ -\ 1} \cdot \frac{|S(B(k))|_p}{v(A(k))\ \sigma(A(k))} \le \frac{p}{p\ -\ 1} \cdot \frac{\psi(p)\ \sigma(B(k))}{v(A(k))\ \sigma(A(k))} \le$$

$$C_2\ \psi_1(p)\ 2^{-k\gamma},$$

where

$$C_2 = \sup_n L(2n)/L(n) < \infty.$$

The proposition of theorem 2 follows after summing over k.

For example, if in addition for $n \ge 16$ and for some $\Delta = const > 0$

$$v(n) \ge (\log n)\ (\log\log n)^{1+\Delta},$$

then

$$|| \sup_n [S(n)/(\sigma(n)\ v(n))]||G(\psi_1) \le C \sup_n ||\ S(n)/\sigma(n)\ ||G(\psi) \cdot (1/\Delta). \tag{6.6}$$

Remark 2. In the probabilistic case $\mu(X) = 1$ or, equally, $\mu(X) < \infty$ the true norming function is $v(n) = (\log\log n)^{1/2}$ for martingales with independent increments, or, in more general case $v(n) = (\log\log n)^{r/2},\ r = 1, 2, \ldots$; see [13], [16], [40]. This is an open question: what is the true norming function $v = v(n)$ in the unbounded case $\mu(X) = \infty$?

6. Applications into the theory of Fourier series.

In this section we intend to obtain the uniform $G(\psi)$ bounds for *maximal function* for the partial sums of Fourier series.

Let $X = [-\pi, \pi],\ \mu(dx) = dx,\ c(n) = c(n, f) =$

$$\int_{-\pi}^{\pi} \exp(inx) f(x) dx, n = 0, \pm 1, \pm 2 \ldots;\ 2\pi s_M[f](x) =$$

$$\sum_{\{n:|n| \le M\}} c(n) \exp(-inx),\ s^*[f] = \sup_{M \ge 1} |s_M[f]|;$$

i.e. in the considered case $T = \{1, 2, 3, \ldots.\}$

Let for some function $\psi \in \Psi\ f(\cdot) \in G(\psi)$ and denote for the values $p > 1$

$$\psi_2(p) = p^4\ \psi(p)/(p\ - 1\ )^2.$$

Theorem 3.

$$||s^*[f]||G(\psi_2) \le C\ ||f||G(\psi). \tag{7.1}$$

Proof is at the same as in section 6; we use at the same partition $W = \{[A(k), A(k+1)\ -\ 1]\} = \{[A(k), B(k)] = \{Q(k)\}\}$ of a view:

$$Q(k) = [A(k), B(k)] = [2^{k\ -\ 1}, 2^k\ -\ 1],\ k = 1, 2, \ldots;$$

but instead the Doobs inequality we use the following estimation: at $p > 1$

$$|s^*[f]|_p \le C\ p^4\ |f|_p/(p\ - 1\ )^2;$$

see, for example, [19], p. 183.

The case of Fourier transform (instead Fourier series), the case of wavelet or Haars series and multidimensional case $X = [-\pi, \pi]^d,\ X = R^d,\ d \ge 2$ may be considered analogously. See, e.g. [23].

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